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SOME PRACTICAL CONSIDERATIONS IN THE APPLICATION OF FINITE SOURCE QUEUEING MODELS

by

Zeev Barzily
Donald Gross
Henry D. Kahn

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1. Introduction

Finite source queueing models (machine repair models) (see [4], pp. 118-125) can be useful for a variety of applications. One area of application is in spares provisioning, where, for a given population of operating units (for example, aircraft engines, computer control modules, or lathes) which stochastically fail and are repairable, it is desired to know how many spare units to have on hand as well as the capacity of the repair facility in order to guarantee a certain service level. Gross, Kahn, and Marsh [5] treat such an application in detail.

In the Gross et al study [5], where it was desired to determine the spares inventory and repair capacity over a changing multi-year planning horizon, three key assumptions were necessary in order to apply the classical queueing theoretic solutions. These assumptions are (i) all units in the population are identical, (ii) units and repair facilities operate continuously, and (iii) the system is always in steady state. In addition, units are assumed to fail according to a Poisson process and repair times

are assumed to be exponentially distributed random variables. Since these latter two assumptions were considered to be fairly realistic in a variety of situations, it is the three assumptions enumerated above which are investigated here.

2. Populations with Unequal Failure Rates

When applying the classic machine repair theory, one must assume all units are identical in order to obtain the steady-state system size probabilities required for determining the system service level for a given number of spares and repair channels. The question naturally arises, is this an adequate assumption? Steady-state probabilities obtained by making this simplifying assumption approximate the true probabilities. It is conceptually possible to model a finite source queue with spares having different failure rates for the calling units, but for a system of any reasonable size the equations become unwieldy.

In this section we study a finite source machine repair model having one machine and one spare. Here we allow the calling units to have different failure rates. In particular, the effect of approximating the steady-state probabilities by assuming that all units have failure rates equal to the average failure rate of the calling units is investigated.

In general, letting M denote the population size, y the number of spares, and c the number of repair channels at the repair facility, the model we work out in detail here is for $M = y = c = 1$. Unit 1 has a Poisson failure rate λ_1 and Unit 2 λ_2. It is assumed that service times are independent exponentially distributed random variables with mean $1/\mu$, and that if Unit i (i=1,2) is operating at time t, the probability it will have failed by time $t + \Delta t$ is $\lambda_i \Delta t + o(\Delta t)$. The following states may be defined for the system.

State	Unit 1	Unit 2
0	Operating	Spare
1	Operating	Repair
2	Spare	Operating
3	Repair	Operating
4	Queue	Repair
5	Repair	Queue

The steady-state probabilities must satisfy the following balance equations:

$$\begin{aligned}
\lambda_1 p_0 &= \mu p_1 \\
(\lambda_1+\mu) p_1 &= \lambda_2 p_2 + \mu p_5 \\
\lambda_2 p_2 &= \mu p_3 \\
(\lambda_2+\mu) p_3 &= \mu p_4 + \lambda_1 p_0 \\
\mu p_4 &= \lambda_1 p_1 \\
\mu p_5 &= \lambda_2 p_3 \\
\sum_{i=0}^{5} p_i &= 1 ,
\end{aligned} \tag{1}$$

where p_i is the steady-state probability that the system is in state i, $i=0,1,\ldots,5$. Solution of the system of Equations (1) yields

$$p_1 = \frac{\lambda_1}{\mu} p_0 , \qquad p_4 = (\lambda_2/\mu)^2 p_0 ,$$

$$p_2 = \frac{\lambda_1(\lambda_1+\mu)}{\lambda_2(\lambda_2+\mu)} p_0 , \qquad p_5 = \frac{\lambda_1\lambda_2}{\mu^2}\left(\frac{\lambda_1+\mu}{\lambda_2+\mu}\right) p_0 ,$$

$$p_3 = \frac{\lambda_1(\lambda_1+\mu)}{\mu(\lambda_2+\mu)} p_0 , \tag{2}$$

$$p_0 = \frac{\mu^2\lambda_2(\lambda_2+\mu)}{\mu^3(\lambda_1+\lambda_2) + \mu^2(\lambda_1+\lambda_2)^2 + 2\lambda_1\lambda_2(\mu\lambda_2+\mu\lambda_1+\lambda_1\lambda_2)} .$$

The measure of system performance that we use here is the steady-state probability that a request for a spare unit is met without delay, and is equivalent to the steady-state probability that a failed unit encounters less than y units in repair or waiting for repair. The probability that a failed unit finds the system in a particular state is different from the arbitrary time probabilities given by Equation Set (2). In common queueing terminology these are termed "arriving customer" probabilities, but in this paper they will be designated "failure point" probabilities since they correspond to the occurrence of component failures. For a finite source-no spares queue (see, e.g., Cooper [3], pp. 82 ff), the "failure point" probabilities are equivalent to the general time probabilities for a finite calling population of one less, but this relationship does not hold for the spares case.

The appropriate failure point probabilities denoted by q_n, as contrasted to the general time probabilities denoted by p_n, may be derived as follows.

Using Bayes' Theorem,

$$q_n = \Pr\{\text{system is in state } n \mid \text{failure about to occur}\}$$

$$= \frac{\Pr\{\text{system is in state } n\}\Pr\{\text{failure about to occur} \mid \text{system in state } n\}}{\sum_n \Pr\{\text{system is in state } n\}\Pr\{\text{failure about to occur} \mid \text{system in state } n\}} .$$

The queueing system considered here may be modeled as a birth-death process with

$$\Pr\{\text{failure in } t,t+\Delta t\} = \lambda_n \Delta t + o(\Delta t) ,$$

where

$$\lambda_n = \begin{cases} \lambda_1 , & n=0,1 \\ \lambda_2 , & n=2,3 \\ 0 , & n=4,5 \end{cases}$$

Thus, for $n=0$,

$$q_0 = \lim_{\Delta t \to 0} \frac{p_0[\lambda_1 \Delta t + o(\Delta t)]}{(p_0+p_1)[\lambda_1 \Delta t + o(\Delta t)] + (p_2+p_3)[\lambda_2 \Delta t + o(\Delta t)]} .$$

Dividing the numerator and denominator by Δt and taking the limit yields

$$q_0 = \frac{\lambda_1 p_0}{\lambda_1(p_0+p_1) + \lambda_2(p_2+p_3)} ,$$

where the p_n are given by Equation Set (2). Similarly,

$$q_1 = \frac{\lambda_1 p_1}{\lambda_1(p_0+p_1) + \lambda_2(p_2+p_3)}$$

$$q_2 = \frac{\lambda_2 p_2}{\lambda_1(p_0+p_1) + \lambda_2(p_2+p_3)}$$

$$q_3 = \frac{\lambda_2 p_3}{\lambda_1(p_0+p_1) + \lambda_2(p_2+p_3)} .$$

The availability, or the probability that a failed unit encounters less than $y=1$ units in the system, is therefore

$$\begin{aligned} A &= q_0 + q_2 \\ &= \frac{\mu(\lambda_1+\lambda_2+2\mu)}{2(\mu^2+\lambda_1\mu+\lambda_2\mu+\lambda_1\lambda_2)} . \end{aligned} \tag{3}$$

For purposes of comparison, consider the $M=1$, $y=1$, $c=1$ queue with exponential service, mean $1/\mu$. It is assumed that the two units in the system have failure rates λ_1 and λ_2 , but that both units operate with the average failure rate $(\lambda_1+\lambda_2)/2$. Thus the probability that a unit operating at time t fails by time $t + \Delta t$ is assumed to be $[(\lambda_1+\lambda_2)/2]\Delta t + o(\Delta t)$. The approximate general time probabilities are then (see [4], p. 123),

$$\hat{p}_1 = \frac{\lambda_1 + \lambda_2}{2\mu} \hat{p}_0 ,$$

$$\hat{p}_2 = \left(\frac{\lambda_1+\lambda_2}{2\mu}\right)^2 \hat{p}_0 ,$$

$$\hat{p}_0 = \frac{\mu^2}{\mu^2 + \mu(\lambda_1+\lambda_2)/2 + (\lambda_1^2+2\lambda_1\lambda_2+\lambda_2^2)/4} ,$$

where

$\hat{p}_i$ = approximate steady-state probability of i units in the system, $i=0,1,2$;

and the approximate "failure point" probabilities are

$$\hat{q}_0 = \frac{\hat{p}_0(\lambda_1+\lambda_2)/2}{\hat{p}_0(\lambda_1+\lambda_2)/2 + \hat{p}_1(\lambda_1+\lambda_2)/2} ,$$

$$\hat{q}_1 = \frac{\hat{p}_1(\lambda_1+\lambda_2)/2}{\hat{p}_0(\lambda_1+\lambda_2)/2 + \hat{p}_1(\lambda_1+\lambda_2)/2} ,$$

where the $\hat{p}_i$ are defined above and

$\hat{q}_i$ = approximate probability that a failed unit finds i units in the system, $i=0,1$.

It follows that the availability for this system, or the failure point probability of zero units in the system, assuming an average failure rate for both units, is

$$\hat{A} = \hat{q}_0$$

or

$$\hat{A} = \frac{2\mu}{2\mu + \lambda_1 + \lambda_2} . \tag{4}$$

The approximate availability given by $\hat{A}$ always underestimates the exact availability A . That is,

$$A \geq \hat{A} ,$$

with equality holding only when $\lambda_1 = \lambda_2$. This may be shown as follows: rewriting Equations (3) and (4) in terms of $\rho_1 = \lambda_1/\mu$ and $\rho_2 = \lambda_2/\mu$ yields

$$A = \frac{2 + \rho_1 + \rho_2}{2(1+\rho_1+\rho_2+\rho_1\rho_2)}$$

$$\hat{A} = \frac{2}{2 + \rho_1 + \rho_2} .$$

It is easy to see that $A \geq \hat{A}$ since

$$A - \hat{A} = \frac{(\rho_1-\rho_2)^2}{2(1+\rho_1+\rho_2+\rho_1\rho_2)(2+\rho_1+\rho_2)} \geq 0 \ . \tag{5}$$

The expression (5) is zero only when $\rho_1 = \rho_2$ or, equivalently, when $\lambda_1 = \lambda_2$. Therefore, the approximate $\hat{A}$ is a lower bound on the exact availability.

Some calculations for A and $\hat{A}$ are shown in Table I for different values of ρ_1 and ρ_2 . In most cases the percent difference between A and $\hat{A}$ is small even though there is a large difference between ρ_1 and ρ_2 . The greatest difference is 27.22% for $\rho_1 = .1$ and $\rho_2 = 2.5$. For cases where both ρ_1 and ρ_2 are less than one, the largest difference is 7.11% for $\rho_1 = .1$ and $\rho_2 = .9$. These results suggest that $\hat{A}$ is a reasonable approximation for the true availability, particularly as the differences in the failure rates will probably be small for most practical cases.

It seems reasonable to assume that the approximate availability, based on using the arithmetic mean of the failure rates, will always be a lower bound on the actual availability for systems involving larger numbers of calling units and more than two failure rates. This is intuitively appealing since it would be expected that, with unequal failure rates, the less reliable units are "down" (i.e., in the service facility) more frequently than the more reliable units. Thus the units that are "up" at a given time tend to be better than average, so that the effective aggregate failure rate for the units in the system is less than the arithmetic average of the failure rates of the units.

3. Noncontinuous Operation

Classical finite source queueing theory assumes the units in the operating population operate continuously (along with the repair facilities/

TABLE I

COMPARISON OF ACTUAL AND APPROXIMATE AVAILABILITY CALCULATIONS

ρ_1 \ ρ_2		.2	.3	.4	.5	.6	.7	.8	.9	1.0	1.5	2.0	2.5
.1	A	.871212	.910973	.811688	.787879	.767045	.748663	.732323	.717703	.704545	.654545	.621212	.597403
	$\hat{A}$	.869565	.909091	.800000	.769231	.740741	.714286	.689655	.666667	.645161	.555555	.487805	.434783
	% dif	.19	.21	1.44	2.37	3.43	4.60	5.83	7.11	8.43	15.12	21.48	27.22
.2	A		.801282	.773809	.750000	.729167	.710784	.694445	.679825	.666667	.616667	.583333	.559524
	$\hat{A}$		.800000	.769231	.740741	.714286	.689655	.666667	.645161	.625000	.540540	.476191	.425532
	% dif		.16	.59	1.23	2.04	2.97	4.00	5.10	6.25	12.30	18.37	23.95
.3	A			.741758	.717949	.697115	.678733	.662393	.647773	.634615	.584615	.551282	.527472
	$\hat{A}$			.740741	.714268	.689655	.666667	.645161	.625000	.606061	.526316	.465116	.416607
	% dif			.14	.51	1.07	1.78	2.60	3.52	4.50	9.97	15.63	21.00
.4	A				.690476	.669643	.651260	.634921	.620301	.607143	.557143	.523810	.500000
	$\hat{A}$				.689655	.666667	.645161	.625000	.606061	.588235	.512820	.454545	.408163
	% dif				.12	.44	.94	1.56	2.30	3.11	7.96	13.22	18.37
.5	A					.645833	.627451	.611111	.596491	.583333	.533330	.500000	.476190
	$\hat{A}$					.645161	.625000	.606061	.588235	.571429	.500000	.444440	.400000
	% dif					.10	.39	.83	1.38	2.04	6.25	11.11	16.00
.6	A						.606618	.590278	.575658	.562500	.512500	.479167	.455357
	$\hat{A}$						.606061	.588235	.571429	.555555	.487805	.434783	.392157
	% dif						.09	.35	.73	1.23	4.82	9.26	13.88
.7	A							.571895	.557276	.544118	.494118	.460784	.436975
	$\hat{A}$							.571429	.555555	.540540	.476190	.425532	.384615
	% dif							.08	.31	.66	3.63	7.65	11.98
.8	A								.540936	.527778	.477778	.444444	.420635
	$\hat{A}$								.540540	.526316	.465116	.416667	.377358
	% dif								.07	.28	2.65	6.25	10.29
.9	A									.513158	.463158	.529825	.406015
	$\hat{A}$									.512820	.454545	.408163	.370370
	% dif									.07	1.86	5.04	8.78

repairmen). In practice, this often is not the case. For example, if we are concerned with provisioning spares for engines belonging to a fleet of aircraft, not all aircraft are always operating. Thus any engine (population component) may operate only a fraction of the time, say α .

There are two possible ways to account for this. One way is to decrease the failure rate appropriately; that is, if λ is the actual failure rate for continuously operating units and each unit operates, on the average, a fraction α , then we use an adjusted failure rate $\lambda' = \alpha\lambda$. For example, if a unit has a mean time between failures (MTBF = $1/\lambda$) of 1000 hours and operates only half the time on the average, we would assume its MTBF to be 2000 hours.

A second way to account for noncontinuous operations is to adjust the population size; that is, use an effective population size of $M' = \alpha M$. In many cases this may actually be closer to reality, as exemplified by the aircraft engine problem. In reality, the situation more closely resembles a population of size αM where each unit operates at full failure rate λ , rather than a population of M where each unit operates at a reduced failure rate $\alpha\lambda$.

When applied to a fleet of gas-turbine engine ships where provisioning for the gas generator component of the engine is desired, both adjustment procedures were compared based on absolute and percentage differences in availability (this example is detailed in Gross, et al [5]). The results given in Table II show that there is very little difference between the methods, even for small M , α , and low availabilities. Even though the population size adjustment may more closely reflect the real situation, the advantage to adjusting the failure rate is that no round off difficulties are met, since failure rate is a continuous quantity while population size is discrete.

4. Transient Effects

In order to account for population and reliability growth over a multi-year planning horizon while utilizing classical steady-state theory,

TABLE II

ADJUSTMENT OF POPULATION SIZE VERSUS ADJUSTMENT OF FAILURE RATE

M	α	c	y	A(αλ)	A(αM)	Δ	%Δ
10	.8	3	3	.955	.956	.001	0.04
		2	2	.802	.805	.003	0.36
	.6	2	2	.878	.881	.003	0.37
	.4	2	2	.940	.942	.002	0.22
		1	1	.637	.660	.023	3.51
	.2	1	1	.813	.827	.014	1.69
50	.8	8	8	.971	.972	.001	0.02
		6	7	.904	.905	.001	0.12
		5	5	.655	.661	.006	0.81
	.6	5	5	.855	.858	.003	0.36
		4	3	.473	.485	.012	2.51
	.4	3	5	.895	.901	.006	0.72
		4	4	.886	.890	.004	0.39
		3	3	.683	.698	.015	2.20
182	.8	10	10	.849	.847	-.002	-0.15
		8	11	.757	.756	-.001	-0.05
		11	8	.667	.665	-.002	-0.24
	.6	6	9	.727	.740	.043	1.78
		7	7	.711	.718	.007	0.92
		9	6	.633	.637	.004	0.64
	.4	5	10	.965	.966	.001	0.05
		7	7	.945	.945	.000	0.00
		10	5	.771	.772	.001	0.04
		5	5	.702	.707	.005	0.75
		7	4	.589	.591	.002	0.33
		5	3	.339	.346	.005	2.02

it was necessary to assume the population reached new steady-state levels instantaneously each year in the planning horizon. For this approximation to be valid, the transient effects must be small after a relatively short period, say three months.

Again, for a small size model (M = y = c = 1) we can analytically work out transient solutions in closed form. The derivation follows.

Assuming both units fail at Poisson rate λ and are repaired exponentially at mean rate μ, the differential difference equations resulting from a birth-death analysis are

$$
\begin{aligned}
p_0'(t) &= -\lambda p_0(t) + \mu p_1(t) \\
p_1'(t) &= -(\lambda+\mu)p_1(t) + \lambda p_0(t) + \mu p_2(t) \qquad (6) \\
p_2'(t) &= -\mu p_2(t) + \lambda p_1(t) \ .
\end{aligned}
$$

Taking Laplace transforms (LT), we get

$$
\begin{aligned}
s\bar{p}_0(s) - p_0(0) &= -\lambda\bar{p}_0(s) + \mu\bar{p}_1(s) \\
s\bar{p}_1(s) - p_1(0) &= -(\lambda+\mu)\bar{p}_1(s) + \lambda\bar{p}_0(s) + \mu\bar{p}_2(s) \\
s\bar{p}_2(s) - p_2(0) &= -\mu\bar{p}_2(s) + \lambda\bar{p}_1(s) \ ,
\end{aligned}
$$

or the following three equations in three unknowns to be solved,

$$
\begin{aligned}
(s+\lambda)\bar{p}_0(s) - \mu\bar{p}_0(s) + 0\bar{p}_0(s) &= p_0(0) \\
-\lambda\bar{p}_0(s) + (s+\lambda+\mu)\bar{p}_1(s) - \mu\bar{p}_2(s) &= p_1(0) \qquad (7) \\
0\bar{p}_0(s) - \lambda\bar{p}_1(s) + (s+\mu)\bar{p}_2(s) &= p_2(0) \ .
\end{aligned}
$$

To solve these, we can use Cramer's rule. Thus it is first necessary to evaluate the determinant of the denominator,

$$
D = \begin{vmatrix} s+\lambda & -\mu & 0 \\ -\lambda & s+\lambda+\mu & -\mu \\ 0 & -\lambda & s+\mu \end{vmatrix}
$$

$$
= (s+\lambda)(s+\lambda+\mu)(s+\mu) - \lambda\mu(s+\lambda) - \lambda\mu(s+\mu) \ .
$$

Expanding and cancelling yields

$$D = s^3 + 2s^2\lambda + 2s^2\mu + s\lambda\mu + s\lambda^2 + s\mu^2$$
$$= s(s^2 + 2s\lambda + 2s\mu + \lambda\mu + \lambda^2 + \mu^2) \;. \tag{8}$$

In order to invert the LT, we would like D to be of the form

$$D = (s+a)(s+b)(s+c) \;,$$

which we obtain by letting

$$a = 0$$
$$b = \lambda + \mu + \sqrt{\lambda\mu}$$
$$c = \lambda + \mu - \sqrt{\lambda\mu} \;;$$

that is, (8) is equivalent to

$$D = s(s + \lambda + \mu + \sqrt{\lambda\mu})(s + \lambda + \mu - \sqrt{\lambda\mu}) \;, \tag{9}$$

which can be verified by expanding and cancelling.

Now if we use the initial condition $p_0(0) = 1$, $p_1(0) = p_2(0) = 0$, that is, if all units are "up," the right-hand side of (7) becomes 1,0,0 and

$$\bar{p}_2(s) = \frac{\begin{vmatrix} s+\lambda & -\mu & 1 \\ -\lambda & s+\lambda+\mu & 0 \\ 0 & -\lambda & 0 \end{vmatrix}}{D} = \frac{\lambda^2}{D} \;. \tag{10}$$

From a table of LT inverses (see, for example, Abromowitz and Stegun [1]),

$$\bar{f}(s) = \frac{1}{(s+a)(s+b)(s+c)} \;, \qquad (a \neq b \neq c)$$

yields

$$f(t) = -\,\frac{(b-c)e^{-at} + (c-a)e^{-bt} + (a-b)e^{-ct}}{(a-b)(b-c)(c-a)} \;,$$

so that

$$p_2(t) = \frac{\lambda^2\left[2\sqrt{\lambda\mu} + (\lambda+\mu-\sqrt{\lambda\mu})e^{-(\lambda+\mu+\sqrt{\lambda\mu})t} - (\lambda+\mu+\sqrt{\lambda\mu})e^{-(\lambda+\mu-\sqrt{\lambda\mu})t}\right]}{2\sqrt{\lambda\mu}(\lambda^2+\lambda\mu+\mu^2)}$$

and

$$\bar{p}_1(s) = \frac{\begin{vmatrix} s+\lambda & 1 & 0 \\ -\lambda & 0 & -\mu \\ 0 & 0 & s+\mu \end{vmatrix}}{D} = \frac{\lambda(s+\mu)}{D} = \frac{s\lambda}{D} + \frac{\mu\lambda}{D} .$$

Using (9) for D and (10) gives

$$\bar{p}_1(s) = \frac{\lambda}{(s+\lambda+\mu+\sqrt{\lambda\mu})(s+\lambda+\mu-\sqrt{\lambda\mu})} + \frac{\mu}{\lambda}\bar{p}_2(s) .$$

Inversion gives $[\bar{f} = 1/(s+a)(s+b) \Rightarrow f = \frac{1}{b-a}(e^{-at} - e^{-bt})]$,

$$p_1(t) = \frac{\lambda}{2\sqrt{\lambda\mu}}\left[e^{-(\lambda+\mu-\sqrt{\lambda\mu})t} - e^{-(\lambda+\mu+\sqrt{\lambda\mu})t}\right] + \frac{\mu}{\lambda}p_2(t) .$$

Rather than solving for $\bar{p}_0(s)$ and inverting, it is easier to use

$$p_0(t) = 1 - p_2(t) - p_1(t) .$$

Now since availability at time t is the conditional probability that a failing unit finds the repair system empty, that is, the other unit "up," we have

$$A(t) = \frac{p_0(t)}{p_0(t) + p_1(t)} = \frac{1 - p_2(t) - p_1(t)}{1 - p_2(t)} . \tag{11}$$

Results can be obtained for an initial condition with both units "down" in a similar fashion and are given by

$$p_0(t) = \frac{\mu^2\left[2\sqrt{\lambda\mu} + (\lambda+\mu-\sqrt{\lambda\mu})e^{-(\lambda+\mu+\sqrt{\lambda\mu})t} - (\lambda+\mu+\sqrt{\lambda\mu})e^{-(\lambda+\mu-\sqrt{\lambda\mu})t}\right]}{2\sqrt{\lambda\mu}(\lambda^2+\lambda\mu+\mu^2)}$$

and

$$p_1(t) = \frac{\lambda}{\mu}p_0(t) + \frac{\mu}{2\sqrt{\lambda\mu}}\left[e^{-(\lambda+\mu-\sqrt{\lambda\mu})t} - e^{-(\lambda+\mu+\sqrt{\lambda\mu})t}\right] .$$

These results were used to produce Figures 1 and 2. Figure 1 shows convergence to steady state for two cases yielding the same steady-state availability, one with large λ and μ and the other with small λ and μ .[1] We note that the large λ,μ case converges much faster, although both converge quite quickly for the "both up" initial condition. This illustrates

[1] λ,μ are given in units per day.

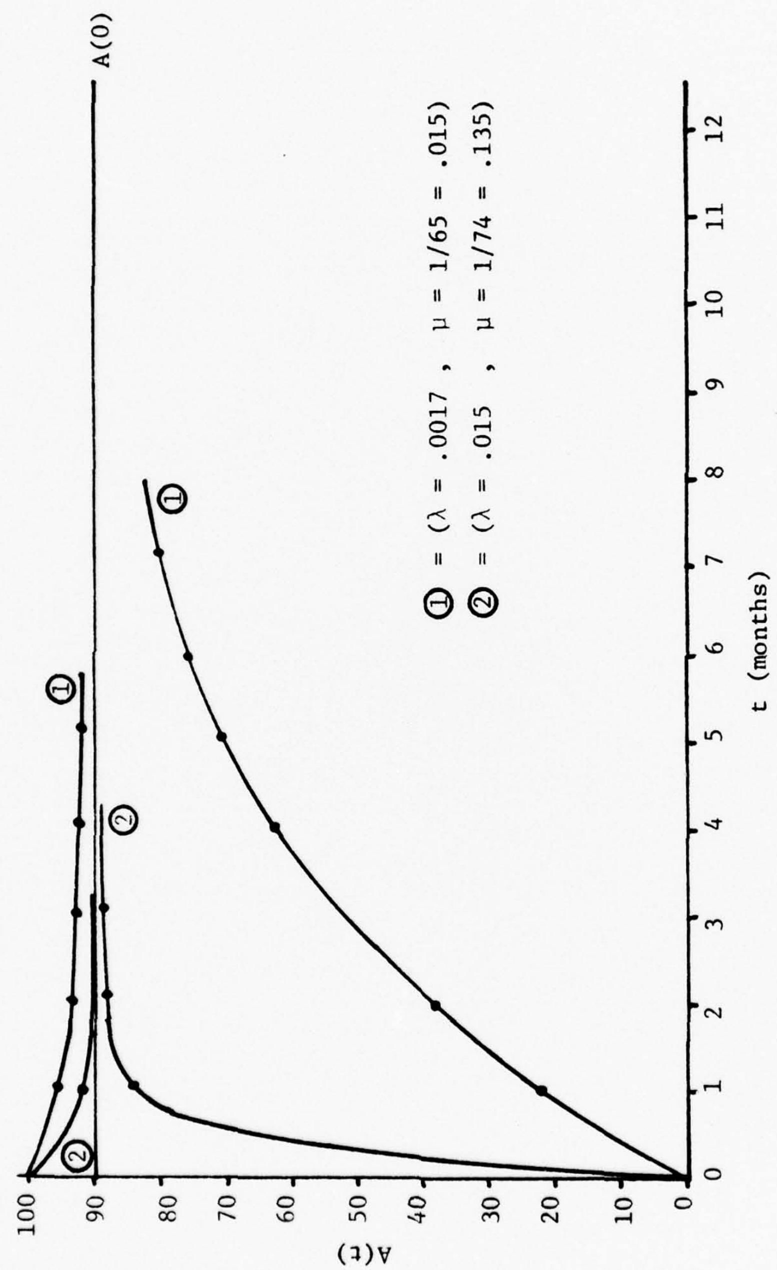

Figure 1.--Convergence to steady state of small scale model (M = y = c = 1).

the two factors affecting convergence to steady state, which are (i) the role of decay of the exponential term and (ii) how far the initial availability is from the steady-state value. For Condition (i) we desire λ and μ large since the transient terms are $e^{-(\lambda+\mu\pm\sqrt{\lambda\mu})t}$, while for Condition (ii) $A(0) = 1$ is much closer to $A(\infty) = .9$ than is $A(0) = 0$.

Figure 2 shows isopercent error at $t=3$ months on the λ-μ plane. The shapes of the curves also illustrate the two effects described above, namely, small percentage error for λ and μ both large, and the effect of initial condition closeness to steady state, since each point on an isopercent error curve has a different $A(\infty) = \mu/(\lambda+\mu)$. The dashed line shows all λ-μ combinations for $A(\infty) = .9$ and we see the percentage errors even for small λ,μ , when $A(0) = 1$, are small.

To obtain similar results for larger population sizes is algebraically untenable. For *transient analysis*, unlike the unequal failure rate problem, making inferences based on the small model is not intuitively as comfortable. Thus we take a different approach to transient analysis for large populations -- *that of the imbedded Markov chain.*

The analysis of the convergence to steady state in Markovian queues can be carried out either through a numerical procedure, as described by Cohen [2], or through an imbedded Markov chain approach, in which the system is observed upon failures of machines. We preferred the imbedded Markov chain approach here because we are interested in the state of the system upon failures of machines, and because by this procedure we are able to obtain some results which we cannot obtain using the other approach.

Let λ_n and μ_n denote the arrival and service rates, respectively, when there are n machines in the system. Thus,

$$\lambda_n = \begin{cases} M\lambda & , \quad n \leq y-1 \\ [M-(n-y)]\lambda , & \quad y \leq n \leq M+y \\ 0 & , \quad \text{otherwise,} \end{cases}$$

and

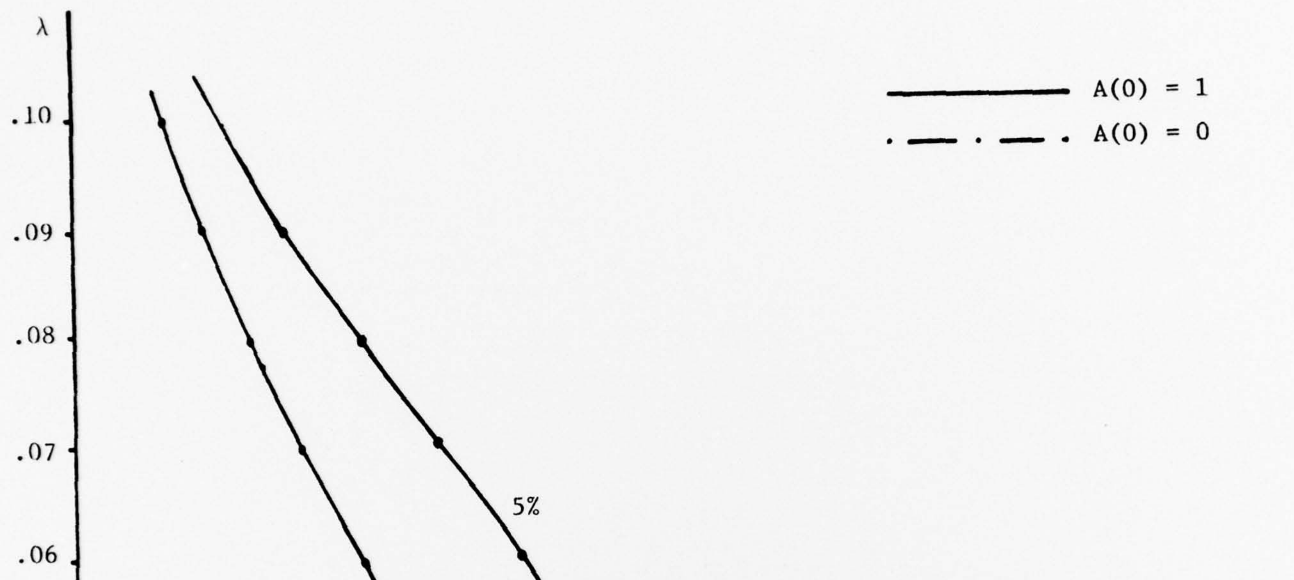

Figure 2.--Isopercent error map, λ-μ plane at t=3 months.

$$\mu_n = \begin{cases} n\mu \ , & 0 \le n \le c \\ c\mu \ , & c < n \ . \end{cases}$$

Denote by $N(m)$ the state of the system upon the mth failure, where $\{N(m)=n\}$, $n=0,1,2,\ldots$, means that the mth failure sees n failed machines in the system. Let

$$p_{n,j} = P[N(m)=j \mid N(m-1)=n] \ ,$$

and denote by A_n and S_n the interfailure time and the time to the completion of the first service, respectively, given that $\{N(m)=n\}$. As both service times and interarrival times are exponential, we note that for $j \le n$,

$$\begin{aligned} p_{n,j} &= \Pr\{\text{n-j+1 service completions before the next arrival} \mid \text{system is in state n}\} \\ &= \Pr\{\text{1 service completion before the next arrival} \mid \text{system is in state n}\} \cdot \\ &\quad \Pr\{\text{n-j service completions before the next arrival} \mid \text{system is in state n-1}\} \\ &= \int_{t=0}^{\infty} P[A_n > t] \, dF_{S_n}(t) \, p_{n-1,j} \ . \end{aligned}$$

For $j = n+1$, there must be an arrival before a service completion. Thus we have,

$$p_{n,j} = \begin{cases} \int_{t=0}^{\infty} P[A_n > t] dF_{S_n}(t) p_{n-1,j} \ , & j=1,2,\ldots,n \ , \ n=0,1,2,\ldots,M+y-2 \\ \int_{t=0}^{\infty} P[S_n > t] dF_{A_n}(t) \ , & j=n+1 \ , \ n=0,1,2,\ldots,M+y-2 \\ 0 \ , & j > n+1 \ . \end{cases} \tag{12}$$

The probabilities $p_{M+y-1,j}$, $j=1,\ldots,M+y-1$, that were not specified in (12), are equal to $p_{M+y-2,j}$ since at least one completion of service should take place before any arrival can occur. We use now the fact that the parameters of A_n and S_n are λ_{n+1} and μ_{n+1} , respectively, and obtain from (12) that

$$p_{n,n+1} = \frac{\lambda_{n+1}}{\lambda_{n+1} + \mu_{n+1}} , \quad n=0,1,\ldots,M+y-2 . \tag{13}$$

Expression (13) can now be used to establish that, for $j=n$ in (12),

$$p_{n,n} = \frac{\mu_{n+1}}{\mu_{n+1} + \lambda_{n+1}} \frac{\lambda_n}{\lambda_n + \mu_n} .$$

Applying the recursive relation in (12) for $j < n$ yields

$$p_{n,j} = \prod_{i=j+1}^{n+1} \frac{\mu_i}{\mu_i + \lambda_i} \frac{\lambda_j}{\lambda_j + \mu_j} , \quad j=0,1,\ldots,n . \tag{14}$$

Now let $N(0)$ denote the initial state of the system and let

$$q_m(i) = P[N(m)=i] , \quad m=0,1,2,\ldots , \quad i=0,1,\ldots,M+y-1 .$$

Denoting by

$$q'_m = \left(q_m(0),q_m(1),\ldots,q_m(M+y-1)\right) , \tag{15}$$

we obtain

$$q'_{m+1} = q'_m P ,$$

where P is the transition matrix $\{p_{i,j}\}$. It is well known then that q'_m has a limit as $m \to \infty$, and let $q' = \lim_{m\to\infty} q'_m$. Denote

$$Q_m(j) = \sum_{i=0}^{j} q_m(i) , \quad m=0,1,2,\ldots , \quad j=0,1,\ldots,M+y-1 ,$$

and

$$Q(j) = \sum_{i=1}^{j} q(i) .$$

A natural initial situation is, of course, all machines up, and one would like to know how this situation would affect the future state vectors. This leads us to the lemma below.

<u>Lemma 1</u>: If $Q_m(j) \geq Q(j)$ for all $j=1,2,\ldots,M+y-1$, then $Q_{m+1}(j) \geq Q(j)$ for all $j=0,1,\ldots,M+y-1$.

Proof: Let

$$L_j = \begin{pmatrix} \ell_{j,0} \\ \ell_{j,1} \\ \cdot \\ \cdot \\ \cdot \\ \ell_{j,M+y-1} \end{pmatrix} , \tag{16}$$

where

$$\ell_{j,i} = \begin{cases} 1 , & i \leq j \\ 0 , & i > j \end{cases} ;$$

hence

$$Q_m(j) = q'_m L_j . \tag{17}$$

Using (15) and the fact that $q'P = q'$, we obtain

$$Q_{m+1}(j) - Q(j) = (q'_m - q')PL_j , \tag{18}$$

where PL_j is a column vector whose ith element equals

$$F_i(j) = \sum_{k=1}^{j} P_{i,k} . \tag{19}$$

From (14) we can see that $p_{n+1,j} < p_{n,j}$ so therefore

$$F_i(j) \geq F_{i+1}(j) . \tag{20}$$

Thus the right-hand side of (18) is nonnegative since the elements in the column vector PL_j are decreasing and the sum of the first k elements of $(q'_m - q')$ is always nonnegative and larger than the magnitude of the k+1st element from the hypothesis of the lemma, $Q_m \geq Q'$.

Since the measure of the system's performance -- the availability -- was defined as

$$A(m) = Q_m(y-1) ,$$

we may conclude from Lemma 1 that if the initial vector satisfies

$$Q_0(j) \geq Q(j) \ , \quad j=1,\ldots,M+y \ ,$$

then

$$A(m) \geq Q(y-1) = A(\infty) \ , \quad m=1,2,3,\ldots \ . \tag{21}$$

The lemma cannot be extended to include $A(m) \geq A(m+1)$. This is because

$$A(m) - A(m+1) = q_m'(I-P)L_{y-1} = q_m(y-1)\frac{\lambda_y}{\lambda_y + \mu_y} - \sum_{k=y}^{M+y-1} q_m(k)F_k(y-1) \ ; \tag{22}$$

hence, the lemma's conditions can be met while Expression (22) is negative. Thus we can only state that the steady-state availability is always less than the transient availabilities.

The decision to purchase spares or to add servers is often made periodically; so it is important to study the relation between time and number of arrivals. It is clear that as $\lambda_n \leq M\mu$ then the time to the kth arrival is stochastically smaller than an Erlang random variable with parameters $M\lambda$ and k $\left(\Gamma(M\lambda,k)\right)$. Furthermore, as the system stays mainly in the states in which $\lambda_n = M\lambda$, it is obvious that the use of $\Gamma(M\lambda,k)$ as the distribution of the time to the kth arrival is highly justified. Nevertheless, if one desires more accurate results for the relations between time and number of arrivals, the results of the following discussion may be applied.

Let $T_n(k)$ denote the time to the kth arrival given $\{N(0)=n\}$. Then

$$P[T_0(k) \leq t] = \int_0^t \lambda_0 e^{-\lambda_0 y} P[T_1(k-1) \leq t-y]dy \ ,$$

$$\begin{aligned} P[T_1(k) \leq t] &= \int_0^t \mu_1 e^{-\mu_1 y} e^{-\lambda_1 y} \int_{u=0}^{t-y} \lambda_0 e^{-\lambda_0 u} P[T_1(k-1) \leq t-y-u]dudy \\ &+ \int_0^t \lambda_1 e^{-\lambda_1 y} e^{-\mu_1 y} P[T_2(k-1) \leq t-y]dy \ , \end{aligned} \tag{23}$$

and for $n > 1$ we obtain

$$P[T_n(k) \leq t] = \int_0^t \mu_n e^{-\mu_n y} e^{-\lambda_n y} P[T_{n-1}(k) \leq t-y]dy$$

$$+ \int_0^t \lambda_n e^{-\lambda_n t} e^{-\mu_n t} P[T_{n+1}(k-1) \leq t-y]dy \ .$$

Denoting

$$P^*_{n,k}(s) = \int_{t=0}^{\infty} e^{-st} \, dP[T_n(k) \leq t]$$

yields after applying Liebnitz's rule, changing order, and evaluating integrals,

$$P^*_{n,k}(s) = \begin{cases} \dfrac{\lambda_0}{\lambda_0+s} P^*_{1,k-1}(s) & , \quad n = 0 \\[2ex] \dfrac{\mu_1}{\mu_1+\lambda_1+s} \dfrac{\lambda_0}{\lambda_0+s} P^*_{1,k-1}(s) + \dfrac{\lambda_1}{\lambda_1+\mu_1+s} P^*_{2,k-1}(s) & , \quad n = 1 \ , \\ & \quad k=1,2,\ldots \\ \dfrac{\mu_n}{\mu_n+\lambda_n+s} P^*_{n-1,k}(s) + \dfrac{\lambda_n}{\mu_n+\lambda_n+s} P^*_{n+1,k-1}(s) & , \quad n > 1 \end{cases} \tag{24}$$

where $P^*_{i,0}(s) = 1$, $i=1,2,\ldots$. From (24) we obtain

$$E[T_n(k)] = \left. \frac{dP^*_{n,k}(s)}{ds} \right|_{s=0} = \begin{cases} \dfrac{1}{\lambda_0} + E[T_1(k-1)] & , \quad n = 0 \\[2ex] \dfrac{1}{\lambda_1+\mu_1} + \dfrac{\mu_1}{\lambda_1+\mu_1} \left(\dfrac{1}{\lambda_0} + E[T_1(k-1)] \right) & \\ \qquad + \dfrac{\lambda_1}{\lambda_1+\mu_1} E[T_2(k-1)] & , \quad n = 1 \\[2ex] \dfrac{1}{\mu_n+\lambda_n} + \dfrac{\mu_n}{\mu_n+\lambda_n} E[T_{n-1}(k)] & \\ \qquad + \dfrac{\lambda_n}{\lambda_n+\mu_n} E[T_{n+1}(k-1)] & , \quad n > 1 \ . \end{cases} \tag{25}$$

We illustrate the convergence to steady state with data from the machine repair problem given in Gross et al [5]. In that study it is assumed that all changes in the parameters of the system occur simultaneously once a year. These changes include the addition of new machines, spares, and servers; an increase in the service rate; and a decrease in the failure rates of all units. Both the units added and the spares added are assumed to be up. Let

$$J(i,\tau) = \min\{k : E[T_i(k) \geq \tau\} . \tag{26}$$

As we observe the system upon failures of machines, we desire to associate the annual points of change in the system's parameters to cumulative counts of failures. We chose $J(0,365)$ to represent the annual number of failures. This assumption and the data from Figure 2 in [5] is presented in Table III.

TABLE III

THE PARAMETERS FOR FIGURE 3

Year	M	c	y	λ^a	$1/\mu^a$	Steady State Availability	$J(0,365)$	Cumulative Count of Failures $TJ(0,365)$
75	10	3	3	0.0015	65.0	0.920	5	5
76	28	5	6	0.0015	62.5	0.920	15	20
77	50	8	8	0.0015	60.0	0.915	26	46
78	82	10	10	0.0013	57.5	0.913	37	73
79	121	12	11	0.0010	55.0	0.905	46	119

[a] λ and μ are shown in units per day.

Figure 3 shows the availability versus time for the data of Table III, assuming an initial condition of all units up, using Equations (13) and (14) to calculate the p_{ij} and then performing successive multiplication with the P matrix; thus yielding q''s , Q's and A's . From

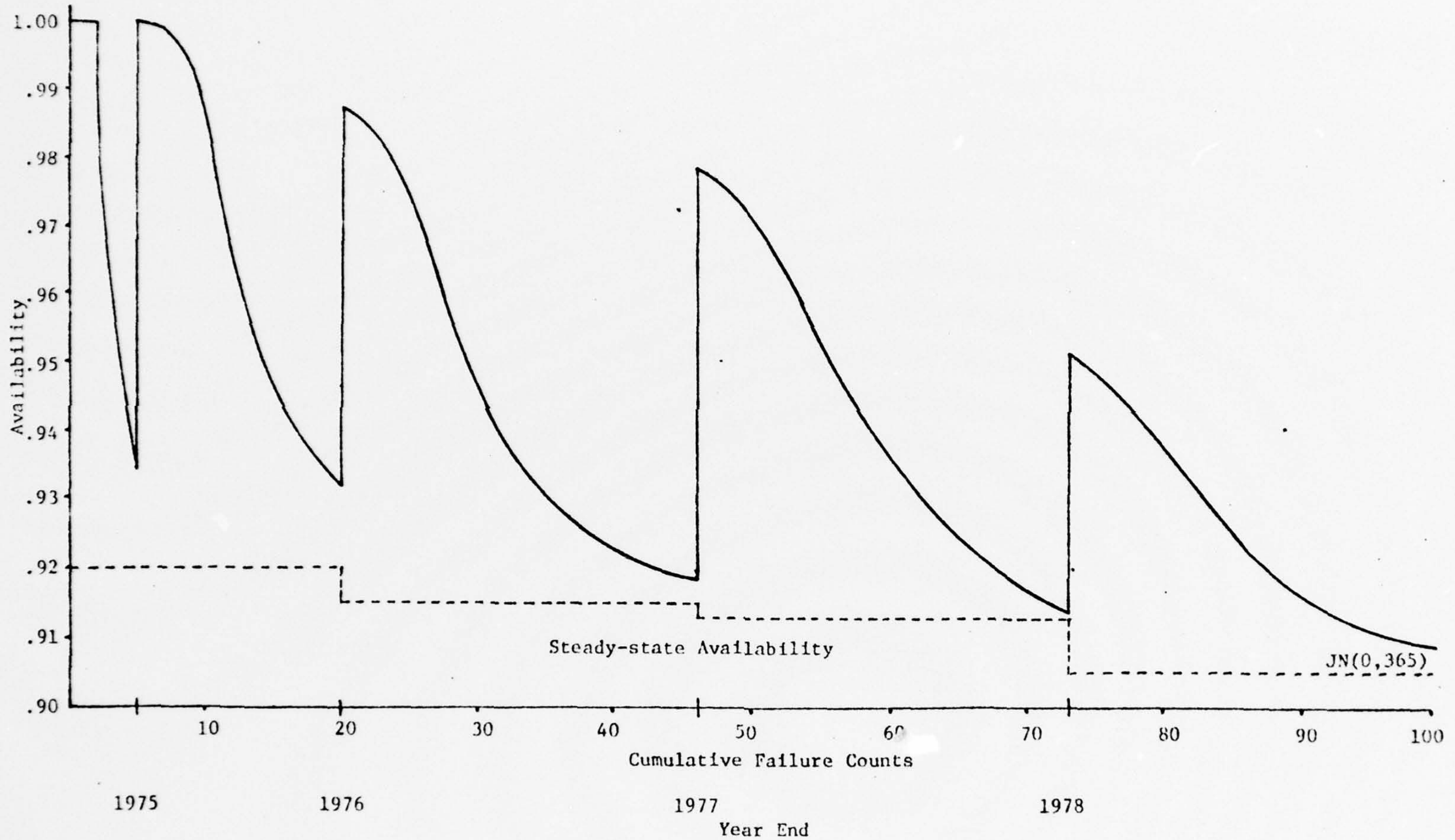

Figure 3.--Availability as a function of cumulative failure counts for the data of Table III.

Figure 3 we can see that when the number of calling machines is small (and therefore $J(0,365)$ is small) then the system's situation is quite far from steady state even after $J(0,365)$ failures. As the number of calling units increases, the availability after $J(0,365)$ failures approaches the steady-state availability.

To obtain some feeling about the sensitivity of the rate of convergence to steady state as a function of λ and μ, we calculate the value of

$$\Delta(m) = \frac{Q_m(y-1) - Q(y-1)}{Q(y-1)} \cdot 100.0 ,$$

for $N(0) = 0$, $m = J(0,90)$, $M = 10$, and $c = y = 3$. Table IV gives Δ, J, and selected steady-state availabilities for a variety of $M\lambda$ versus $c\mu$ cases. The display of the table is similar to that of Figure 2 in that small values of $M\lambda$ are at the bottom and small values of $c\mu$ are at the left. We can see the same type of effect as caused by the $A(0) = 1$ curve of Figure 2, although the "belly" is reached for much higher $M\lambda$; that is, for fixed $c\mu$, the errors increase up to a certain value of $M\lambda$ and then start to decrease. Again we see the same two effects as in Figure 2; namely, the influence of the proximity of the starting availability to the steady-state value, and the damping effect of large $M\lambda$ and $c\mu$ on the transient terms (hence the "belly" situation). Selected steady-state availabilities are shown which point out that in the realistic availability range (greater than 0.8, for example) the percent errors from steady state after three months are quite small unless both $M\lambda$ and $c\mu$ are very low.

TABLE IV

VALUES OF $\begin{bmatrix} \Delta(J(0,90)) \\ J(0,90) \\ A(\infty) \end{bmatrix}$ AS A FUNCTION OF $M\lambda$ AND $c\mu$

$M\lambda$ \ $c\mu$	0.03	0.06	0.09	0.12	0.15	0.18	0.21	0.24	0.27	0.30
0.0625	94. 15	44.8 17	26.10 19	16.75 21	11.27 23	7.73 25	5.29 27	2.79 30	1.85 32	1.17 34
0.0525	389. 14	59.7 17	34.59 19	21.73 21	14.10 23	9.19 25	5.89 27	3.66 29	2.70 30	1.53 32
0.0425	444. 14	139.3 16	65.40 18	35.52 20	20.32 22	11.68 24	8.10 25	4.34 27	2.23 29	1.32 30
0.145	2691. 10	242.3 11	57.51 12	17.82 13	7.48 13	3.31 13	1.52 13	.72 13	.36 13 .757	.18 13 .804
0.130	1946. 10	176.5 11	50.78 11	14.88 12	5.99 12	2.57 12	1.16 12	.54 12 .753	.27 12 .805	.14 12 .845
0.115	2144. 9	157.0 10	42.06 10	14.82 10	4.51 11	1.88 11	.84 11 .747	.39 11 .807	.19 11 .850	.10 11 .882

TABLE IV--continued

$M\lambda$ \ $c\mu$	0.03	0.06	0.09	0.12	0.15	0.18	0.21	0.24	0.27	0.30
0.100	2034. 8	128.0 9	32.44 9	11.03 9	4.33 9	1.88 9 .740	.87 9 .810	.43 9 .857	.23 9 .891	.12 9 .915
0.085	1088. 8	94.5 8	23.02 8	7.63 8	2.97 8 .730	1.29 8 .813	.60 8 .866	.30 8 .901	.16 8	.04 8
0.070	741. 7	62.2 7	14.79 7	4.84 7 .714	1.89 7 .818	.83 7 .878	.40 7 .915	.20 7	.11 7	.06 7
0.055	566. 5	47.6 5	12.21 5 .688	4.42 5 .825	1.92 5 .893	.94 5 .931	.500 5	.28 5	.17 5	.10 5
0.040	235. 4	24.0 4 .635	6.56 4 .836	2.49 4 .915	1.13 4	.58 4	.32 4	.19 4	.12 4	.08 4
0.025	71. 3 .508	9.3 3 .857	2.76 3 .945	1.11 3	.54 3	.29 3	.17 3	.11 3	.07 3	.05 3
0.010	9. 1 .915	1.5 1	.49 1	.22 1	.12 1	.07 1	.04 1	.03 1	.02 1	.02 1

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